Finding signed Areas and Volumes inspired by Technology

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Abstract

We describe how we can find the singed area bounded by a parametric curve with respect to a slanted line, $y = mx + b$, in two dimensions. We extend this idea to the corresponding setting in three dimensions. These are special cases of Green's Theorem and the Divergence Theorem, but the ideas and methods introduced here are accessible to a wider audience.

1 Introduction

In a calculus textbook (see [1] or [2]), the Riemann integral $\int_a^b f(x)dx$ for a continuous function f over the interval $[a, b]$ represents the net area of the region bounded by the curve $y = f(x)$, two vertical lines $x = a$ and $x = b$ (both perpendicular to the x-axis), and the x-axis. In this paper, we discuss ways of finding the area bounded by a parametric curve $[x(t), y(t)]$, a slanted line $y = mx + b$, and the perpendiculars to the line from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$ in two dimensions. Since we replace the graph of a function $y = f(x)$ with a parametric curve, the sign of the bounded area depends on the direction of the given parametric curve. Throughout this paper, we will use the terminology of *finding the net signed area (or simply area) bounded by a parametric curve $[x(t), y(t)]$ with respect to a slanted line $y = mx + b$* without causing any confusion. We generalize the idea from two dimensions (2-D) to three dimensions (3-D) and use similar terminology throughout the paper.

In Section 1, we illustrate how technology has inspired us to conceive formulae in two dimensions. In Section 2, we generate the 2-D formulae to corresponding ones in 3-D. In Section 3, we describe a special method of finding the volume bounded by a surface satisfying the form $F(x, y, z) = 0$ and a general plane. We do not require readers to have knowledge of Green's Theorem or the Divergence Theorem. However, we remind those readers who are familiar with Green's Theorem and the Divergence Theorem that the results derived in this paper are consistent with those of Green's Theorem and the Divergence Theorem.

1.1 Motivation

We intend to find the area of one of the regions bounded by the curve $C : x^2 + y^2 + 2.22x - 0.1y - 0.529 = 0$ and the line $\overleftrightarrow{EF} : y = 0.9802x + 1.866$, which is shown in Figure 1. We note that the curve C and the line $\overleftrightarrow{EF}$ were drawn arbitrarily employing Dynamic Geometry software [ClassPad Manager]. The line $\overleftrightarrow{EF}$ divides the circle into two portions, we select the smaller portion (see Figure 1) for demonstration. We select such C because if we were to apply traditional techniques using either vertical or horizontal partitions (dx or dy respectively), we would need to divide the area into two sub-areas since a vertical or horizontal line intersects C at two different intersections. However, if we think of EF as our new x-axis and choose one line that is perpendicular to EF as our new y-axis, then we avoid multiple partitions and can apply a numerical approximation technique to approximate the Riemann integral for the enclosed region before analytically finding the bounded area.
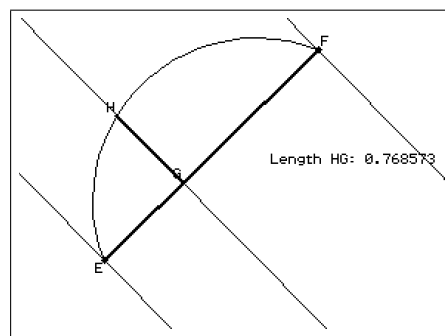

Figure 1. Motivation

We will use ClassPad (see [3]) to describe how we apply the Trapezoidal Rule to approximate the area of the region.

- We note the distance of EF equals 2.444009. Subdivide EF into 19 panels: $2.444009/19 = 0.1286320526$.

- Each subinterval is of length 0.1286320526.

- The length of HG is tabled below:

$$
\begin{bmatrix}
0 \\
0.2338395223 \\
0.392711369 \\
0.511848075 \\
0.6039579856 \\
0.6753060238 \\
0.7294542441 \\
0.7685729318 \\
0.7940051793 \\
0.8065384591 \\
0.8065384591 \\
0.7940051793 \\
0.7685729318 \\
0.7294542441 \\
0.6753060238 \\
0.6039579856 \\
0.511848075 \\
0.392711369 \\
0.2338395223 \\
0
\end{bmatrix}
$$

Table 1. The heights of HG.

- We calculate the area of each trapezoidal region and table them below. Note that column A represents the height HG and column B represents the area of each trapezoidal region. For example the cell $B1 = \dfrac{(A1 + A2) \cdot (0.1286320526)}{2}$ is the area of the first trapezoid.

	A	B	C	D	E	F	G	H	I	J	K	L
1	0	0.0150	1.4191									
2	0.2338	0.0403										
3	0.3927	0.0582										
4	0.5118	0.0718										
5	0.6040	0.0823										
6	0.6753	0.0903										
7	0.7295	0.0963										
8	0.7686	0.1005										
9	0.7940	0.1029										
10	0.8065	0.1037										
11	0.8065	0.1029										
12	0.7940	0.1005										
13	0.7686	0.0963										
14	0.7295	0.0903										
15	0.6753	0.0823										
16	0.6040	0.0718										
17	0.5118	0.0582										
18	0.3927	0.0403										
19	0.2338	0.0150										
20	0	0										
21												
22												
23												

`=(A1+A2)·0.1286320526/2`

`B1 0.01503962887`

Table 2. Trapezoidal Sum

- We sum up the values in column B and save it in cell $C1 = 1.4191$, which is the approximation of the bounded area.

- The video clip and eActivity using ClassPad (see [3]) describing this example can be found in [5] and [6] respectively.

1.2 Theoretical Approach

A curve $\mathbf{w}(t)$ is smooth if $\mathbf{w}'(t)$ is continuous and never equal to the zero vector. We first integrate a smooth curve C of the form $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \leq t \leq t_2$, with respect to a line $L : y = mx$ through the origin. We remark that the curve C will travel from $t = t_1$ to $t = t_2$; this becomes important when we interpret the area, which will have positive or negative value. For example, we sketch the of the curve C (that is traveling from P to Q), and the line segment AB, which is $y = mx$,

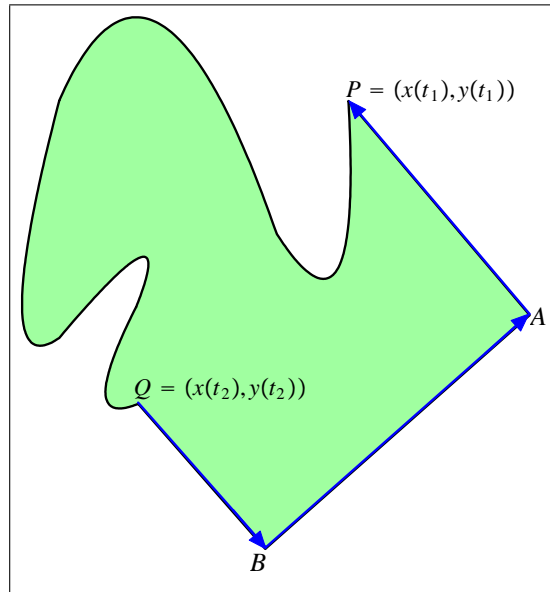in Figure 2 below. We note that PA and QB are both perpendicular to the line segment AB.

Figure 2. Area boundedby APQBA

Let $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and $\mathcal{B}_2 = \{\mathbf{c}_1, \mathbf{c}_2\}$ be another basis for $\mathbb{R}^2$, where $\mathbf{c}_1 = (\cos\theta, \sin\theta)$ is a unit direction vector of the line L and $\mathbf{c}_2 = (-\sin\theta, \cos\theta)$ is a unit vector perpendicular to $\mathbf{c}_1$ with $\theta = \tan^{-1} m$.

We first express $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ as a vector relative to the basis $\mathcal{B}_2$. In other words, we need to discover $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ so that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = p(t)\mathbf{c}_1 + q(t)\mathbf{c}_2$, or

$$
\begin{aligned}
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= [\mathbf{c}_1\ \mathbf{c}_2] \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}.
\end{aligned}
\tag{1}
$$

This implies that

$$
\begin{aligned}
\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&\quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&\quad \begin{bmatrix} x(t)\cos\theta + y(t)\sin\theta \\ -x(t)\sin\theta + y(t)\cos\theta \end{bmatrix}.
\end{aligned}
\tag{2}
$$

The integral of $[x(t), y(t)]$ with respect to a line $L : y = mx$ through the origin is

$$
\begin{aligned}
A &= \int_{t_1}^{t_2} q(t)p'(t)dt \\
&= \int_{t_1}^{t_2} \left(-x(t)\sin\theta + y(t)\cos\theta\right)\left(x'(t)\cos\theta + y'(t)\sin\theta\right)dt \\
&= \cos^2\theta \int_{t_1}^{t_2} \left(-x(t)\frac{\sin\theta}{\cos\theta} + y(t)\frac{\cos\theta}{\cos\theta}\right)\left(x'(t)\frac{\cos\theta}{\cos\theta} + y'(t)\frac{\sin\theta}{\cos\theta}\right)dt \\
&= \frac{1}{1+\tan^2\theta} \int_{t_1}^{t_2} \left(-x(t)\tan\theta + y(t)\right)\left(x'(t) + y'(t)\tan\theta\right)dt \\
&= \frac{1}{1+m^2} \int_{t_1}^{t_2} \left(-x(t)m + y(t)\right)\left(x'(t) + y'(t)m\right)dt
\end{aligned}
\tag{3}
$$

Next, integrate a parametric curve $\mathbf{w}(t) = [x(t), y(t)]$, $t_1 \le t \le t_2$, with respect to a line $L : y = mx + b$ (let $\theta = \tan^{-1} m$). By shifting the bounded region vertically by b, we can then apply the same concept above to find the coordinate vector $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ of $\begin{bmatrix} x(t) - 0 \\ y(t) - b \end{bmatrix}$ relative to $\mathcal{B}_2$ below.

$$
\begin{aligned}
\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x(t) - 0 \\ y(t) - b \end{bmatrix} \\
&= \begin{bmatrix} x(t)\cos\theta + (y(t) - b)\sin\theta \\ -x(t)\sin\theta + (y(t) - b)\cos\theta \end{bmatrix}.
\end{aligned}
\tag{4}
$$

Therefore, the integral of $\mathbf{w}(t) = [x(t), y(t)]$, $t_1 \le t \le t_2$, with respect to the line $L : y = mx + b$ (let $\theta = \tan^{-1} m$) is

$$
\begin{aligned}
A &= \int_{t_1}^{t_2} q(t)p'(t)dt \\
&= \int_{t_1}^{t_2} \left(-x(t)\sin\theta + (y(t) - b)\cos\theta\right)\left(x'(t)\cos\theta + y'(t)\sin\theta\right)dt \\
&= \cos^2\theta \int_{t_1}^{t_2} \left(-x(t)\frac{\sin\theta}{\cos\theta} + (y(t) - b)\frac{\cos\theta}{\cos\theta}\right)\left(x'(t)\frac{\cos\theta}{\cos\theta} + y'(t)\frac{\sin\theta}{\cos\theta}\right)dt \\
&= \frac{1}{1+\tan^2\theta} \int_{t_1}^{t_2} \left(-x(t)\tan\theta + y(t) - b\right)\left(x'(t) + y'(t)\tan\theta\right)dt \\
&= \frac{1}{1+m^2} \int_{t_1}^{t_2} \left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt.
\end{aligned}
\tag{5}
$$

To restate what we have just proved:

Theorem 1 *Let C be the smooth curve $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \le t \le t_2$. Let R be the region bounded by C, the line $y = mx + b$, and the perpendiculars to the line from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$. Then the area of R is given by*

$$
\frac{1}{1+m^2} \int_{t_1}^{t_2} \left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt.
\tag{6}
$$

Green's Theorem in our discussion can be summarized as follows:

Theorem 2 *Let C be the smooth curve $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \leq t \leq t_2$. Let R be the region bounded by C, the line $y = mx + b$, and the perpendiculars to the line from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$. We denote the counterclockwise boundary curve of R by ∂R. If P and Q are scalar fields with continuous partial derivatives satisfying $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then*

$$
\begin{aligned}
\int_{\partial R} P\,dx + Q\,dy &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \frac{-1}{1+m^2} \int_{t_1}^{t_2} (-x(t)m + y(t) - b)\,(x'(t) + y'(t)m)\,dt,
\end{aligned}
\tag{7}
$$

Note. *For convenience, we choose $P(x, y) = -\frac{y}{2}$ and $Q(x, y) = \frac{x}{2}$.*

Remark 3 *Observe Theorem 1 proffers an explicit formula for finding the area enclosed by a parametric closed curve. Suppose the curve C is a semicircle described by the thick curve, $[\cos t, \sin t]$, traveling from D to E, where $t \in [0, \pi]$. The line $L = \overleftrightarrow{AB}$ is $y = x - 4$, as shown in Figure 2.*
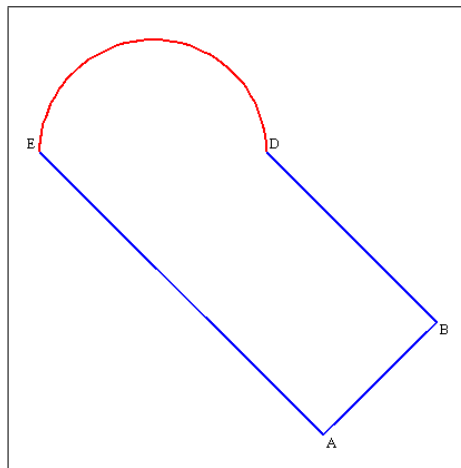

Figure 3. Area with direction

The area of the the region R bounded by C, the line L, and the perpendiculars to the line L from D to E is enclosed by the closed curve $D \to E \to A \to B \to D$. This area is the area of the semicircle plus the area of the trapezoid $DEAB$. Since the closed curve is traversed counterclockwise, (6) gives us a negative value of $-\frac{\pi}{2} - 4$, and therefore the area is $\frac{\pi}{2} + 4$.

For the remainder of the paper, the word "area" represents a net signed area. The area enclosed by a counterclockwise curve is negative and the area enclosed by a clockwise curve is positive due to the construction of $\mathbf{c}_1$ in (1) in section 1.2.

Remark 4 *Set $m = b = 0$ in (6), then*

$$
\int_{t_1}^{t_2} y(t)x'(t)dt,
\tag{8}
$$

as expected. In addition, (7) also provides the signed area enclosed by a curve $[x(t), y(t)]$, a line L, and the perpendiculars to the line L from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$.

Corollary 5 *Let C be a closed smooth curve traveling only in one direction, then (6) produces the area bounded by the closed curve which is independent of the location of the line L.*

Remark 6 *Corollary 5 and the example in Remark 3 coincide with Green's Theorem in the sense that the area enclosed by a closed piecewise smooth curve can be obtained by utilizing contour integrals along the boundary curve traversed in one direction.*

1.3 Examples

Using Theorem 1, we can find the area bounded by two parametric curves without utilizing Green's Theorem.

Example 7 *Given two curves, a cardioid $C_1 = [2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t)]$ and a circle $C_2 = [0.5 + 3\cos(t), 0.5 + 3\sin(t)]$, where $t \in [0, 2\pi]$.*
(1) Find the area of the region that is inside the cardioid C_1 and exterior to the circle C_2. (See Figure 3 below.)
(2) Find the area of the region that is inside both C_1 and C_2.
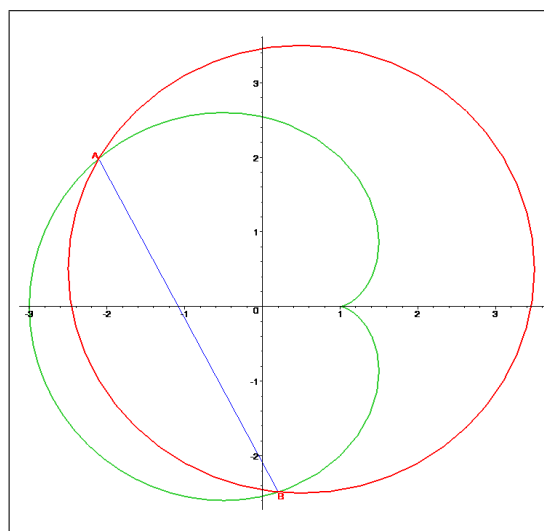

Figure 4. Area between two parametric curves

Part (1):
Step 1. First, we find the intersections of these two curves at

$$A = (-2.108615230, 1.981595956) \text{ and } B = (.2046062939, -2.485421672),$$

respectively.
Step 2. We calculate the area bounded by C_1 and line segment AB. We integrate along C_1 counterclockwise from A (t = 2.574088482) to B (t = 4.402664587) by applying (6) to compute the area bounded by C_1 and the line equation AB :

$$y = -1.931x - 2.09,$$

and obtain

$$\frac{1}{1+m^2}\int_{t_1}^{t_2}(-x(t)m+y(t)-b)\left(x'(t)+y'(t)m\right)dt \tag{9}$$

$$= \frac{1}{1+(-1.931)^2}\int_{2.574088482}^{4.402664587}\begin{array}{l}((\cos 2t - 2\cos t)(-1.931)+2\sin t - \sin 2t + 2.09)\cdot\\(-2\sin t + 2\sin 2t + (2\cos t - 2\cos 2t)(-1.931))\end{array}dt$$

$$= -7.538433161.$$

Note that the answer above is negative since we integrate along a counterclockwise path.
Step 3. We next calculate the area bounded by C_2 and the line segment AB. We integrate along C_2 clockwise from B ($t = 4.613764607$) to A ($t = 2.625063229$) by applying (6) to compute the area bounded by C_2 and the line equation AB :

$$y = -1.931x - 2.09,$$

and obtain

$$\frac{1}{1+m^2}\int_{t_1}^{t_2}(-x(t)m+y(t)-b)\left(x'(t)+y'(t)m\right)dt \tag{10}$$

$$= \frac{1}{1+(-1.931)^2}\int_{4.613764607}^{2.625063229}\begin{array}{l}((-0.5-3\cos t)(-1.931)+0.5+3\sin t + 2.09)\cdot\\(-3\sin t + 3\cos t)(-1.931))\end{array}dt$$

$$= 4.836961037.$$

Therefore, the area bounded between C_1 and C_2 inside the cardioid C_1 and exterior to the circle C_2 is 2.701472124.

Part (2): We first find the area of the cardioid of C_1 by applying equation (8), which yields 18.84955592.
Next we subtract the answer of part 1 from the area of the cardiod C_1. Our result for part (2) is 16.14808380.
The Maple worksheet for this example can be found in [7].

Remark 8 *Let C_1^* be the path traversed along C_1 counterclockwise from A to B and C_2^* be the path traversed along C_2 clockwise from B to A. Using Green's Theorem on the curve $C = C_1^* \cup C_2^*$, we obtained* 2.701472124 *for the area of the region enclosed by C.*

$$\frac{1}{2}\int_{C_1^*}(x(t)y'(t)-y(t)x'(t))\,dt + \frac{1}{2}\int_{C_2^*}(x(t)y'(t)-y(t)x'(t))\,dt$$

$$= 9.955568535 - 7.254096410 = 2.701472125.$$

Example 9 *Let the curve C be an ellipse given by $[A\cos t, B\sin t]$, where $t \in [0, 2\pi]$, and a line L : $y = mx + b$. We obtain $AB\pi$ for the area of the region inside the ellipse by applying (6), regardless of whether the line L intersects the curve C.*

Another example below shows that if a curve traverses in only one direction, we will obtain the area enclosed by the curve by applying (6).

Example 10 *Let $r(t) = \sin 3t$, $f(t) = r(t) \cdot \cos t$, $g(t) = r(t) \cdot \sin t$, the line $L = \overleftrightarrow{DE}$ be $y = -0.9x - 1.378$. If C is the curve $[f(t), g(t)]$, where $t \in [0, \frac{\pi}{2}]$. Find the area bounded by C with respect to the line L.*
Sketch the graph of C, line $L = \overleftrightarrow{DE}$, and perpendiculars AE and BD, from $A = [f(0), g(0)]$ and $B = [f(\frac{\pi}{2}), g(\frac{\pi}{2})]$ to L, respectively, in Figure 5 below. Find the areas of two portions, one being the loop going from A back to A in a counterclockwise direction and the other enclosed by $ABDEA$ in a clockwise direction. The area of the first portion is

$$\int_0^{\frac{\pi}{3}} g(t) f'(t) dt = -\frac{\pi}{12} = -0.2617993878.$$
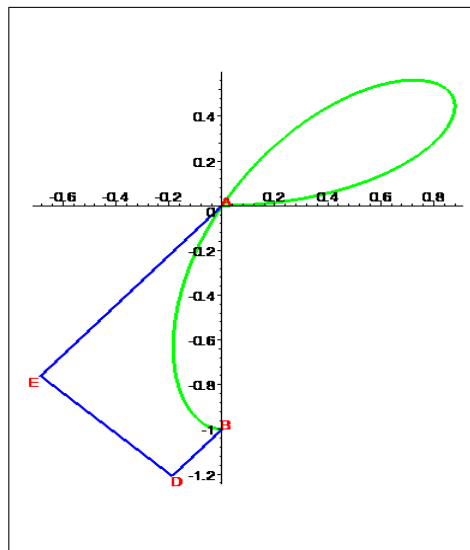

Figure 5. Area of a parametric curve with respect to a slanted line

We apply formula (6) to compute the area for the second portion

$$\frac{1}{1+m^2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(-x(t)m + y(t) - b \right) \left(x'(t) + y'(t)m \right) dt = 0.3056748917.$$

Therefore, the area of the region bounded by C with respect to the line L is 0.5674742795. The corresponding Maple worksheet can be found in [8].

Example 11 *Let $r(t) = \sin 3t$, $f(t) = r(t) \cdot \cos t$, $g(t) = r(t) \cdot \sin t$. Furthermore, define*

$$\begin{aligned} F(t) &= f(t - \frac{\pi}{6}) \text{ and} \\ G(t) &= g(t + \frac{\pi}{6}). \end{aligned}$$

Find the area enclosed by $[F(t), G(t)]$, where $t \in [0, \pi]$. Sketch the graph of $[F(t), G(t)]$ as follows:
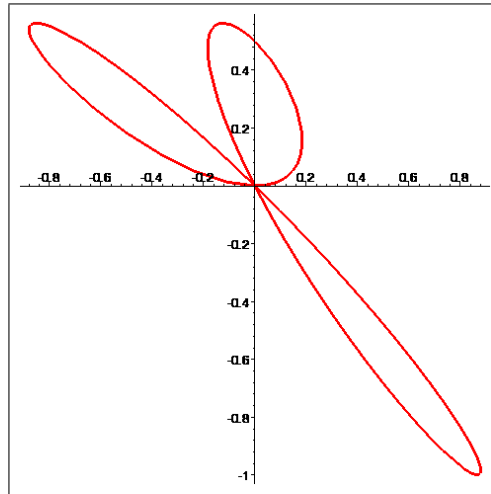

Figure 6. Area of a parametric curve which travels in only one direction

By Corollary 5, we can apply (8) in Remark 4 to obtain the value $\frac{\pi}{8}$. Note that since the curve travels only in one (clockwise) direction (we leave this verification to the reader), the value represents the area enclosed by the curve. The Maple worksheet for this example can be found in [9].

2 Extending ideas from 2-D to 3-D

Suppose $x(s, t)$, $y(s, t)$, and $z(s, t)$ are scalar functions, and $D \subseteq \mathbb{R}^2$. The set

$$\mathbf{w}(s, t) = \{[x(s, t), y(s, t), z(s, t)] \in \mathbb{R}^3 : (s, t) \in D\}$$

is called a parametric surface or simply a surface. The parametric surface is called *smooth* if the tangent vectors in the s and t directions satisfy

$$\mathbf{w}_s \times \mathbf{w}_t \neq (0, 0, 0) \text{ throughout its domain.} \tag{11}$$

For the rest of paper, we will assume a surface is *orientable*; i.e., it is possible to choose a unit normal vector $\mathbf{n}$ at every point continuously over the surface.

We want to integrate a smooth parametric surface $\mathbf{w}(s, t) = [x(s, t), y(s, t), z(s, t)]$, $t_1 \leq t \leq t_2$, $s_1 \leq s \leq s_2$, with respect to a plane P passing through the origin and spanned by vectors $\mathbf{u}$ and $\mathbf{v}$; i.e., $P = \{\mathbf{0} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$.

Let $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and $\mathcal{B}_2 = \{\frac{\mathbf{p}_1}{\|\mathbf{p}_1\|}, \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|}, \frac{\mathbf{n}}{\|\mathbf{n}\|}\}$ be another orthonormal basis for $\mathbb{R}^3$. To form $\mathcal{B}_2$:

Case 1: If $\mathbf{u} \perp \mathbf{v}$, then we choose $\mathbf{p}_1 = \mathbf{u}$, $\mathbf{p}_2 = \mathbf{v}$, and $\mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2$.

Case 2: If $\mathbf{u}$ is not perpendicular to $\mathbf{v}$, then $\mathbf{p}_1$ is the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$, $\mathbf{p}_2 = \mathbf{u} - \mathbf{p}_1$, and $\mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2$ is the normal vector for the plane P.

First we need to find the vector $\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix}$ so that

$$[x(s,t), y(s,t), z(s,t)] = p(s,t)\frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} + q(s,t)\frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} + r(s,t)\frac{\mathbf{n}}{\|\mathbf{n}\|} \text{ or} \tag{12}$$

$$\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix} \begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}. \tag{13}$$

The integral of the surface $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \le t \le t_2$, $s_1 \le s \le s_2$, with respect to P is

$$A = \iint r(s,t)dpdq \tag{14}$$

$$= \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(s,t) \left| \begin{bmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{bmatrix} \right| ds dt.$$

We are ready to integrate a smooth parametric surface $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \le t \le t_2$, $s_1 \le s \le s_2$, with respect to a general plane $P = \{\mathbf{d} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ that does not pass through the origin and has only one z-intercept $\mathbf{d}$. In other words, P is the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, passing through $\mathbf{d} = (0, 0, d_3)$.

Using the same notation as we did previously, we adjust (13) by replacing $\mathbf{w}$ with $\mathbf{w} - \begin{pmatrix} 0 \\ 0 \\ d_3 \end{pmatrix}$

(by doing this, we are shifting the surface $\mathbf{w}$ down vertically by d_3). We have

$$\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) - d_3 \end{bmatrix}. \tag{15}$$

Therefore, the **volume** bounded by the parametric surface $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \le t \le t_2$, $s_1 \le s \le s_2$, with respect to a general plane $P = \{\mathbf{d} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ is

$$A = \iint r(s,t)dpdq \tag{16}$$

$$= \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(s,t) \left| \begin{bmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{bmatrix} \right| ds dt.$$

We next verify that our 3-D formula does reduce to the corresponding one in 2-D. We consider $\mathbf{w}(t) = [x(t), y(t), z(t)]$, $t_1 \le t \le t_2$.

The plane equation $P = \{\mathbf{d} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ becomes a line $L = \{\mathbf{d} + a\mathbf{u} : a \in \mathbb{R}\}$ when $b = 0$, and the slope m of this line is $\frac{u_2}{u_1}$, where $\mathbf{u} = (u_1, u_2)$. Let $\theta = \tan^{-1}(m)$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{e}_1}{\|\mathbf{u}\| \|\mathbf{e}_1\|} = \frac{\mathbf{u} \cdot (1,0)}{\|\mathbf{u}\|} = \frac{u_1}{\|\mathbf{u}\|},$$

$$\sin\theta = \cos(\frac{\pi}{2} - \theta) = \frac{\mathbf{u} \cdot \mathbf{e}_2}{\|\mathbf{u}\| \|\mathbf{e}2\|} = \frac{\mathbf{u} \cdot (0,1)}{\|\mathbf{u}\|} = \frac{u_2}{\|\mathbf{u}\|}.$$

In this case, we use $\mathbf{p}_1 = \mathbf{u} = (u_1, u_2)$, and $\mathbf{n} = (-u_2, u_1)$ (note that $\mathbf{p}_1 \perp \mathbf{n}$ and $\|\mathbf{n}\| = \|\mathbf{p}_1\| = \|\mathbf{u}\|$). We first find the **coordinate vector** $[p(t), q(t)]$ of $[x(t), y(t)]$ with respect to the basis $\{\frac{\mathbf{p}_1}{\|\mathbf{p}_1\|}, \frac{\mathbf{n}}{\|\mathbf{n}\|}\}$, where $\frac{\mathbf{p}_1}{\|\mathbf{p}_1\|}$ is an unit direction vector of the line $L = \{\mathbf{d} + a\mathbf{u} : a \in \mathbb{R}\}$. Let $\mathbf{d} = (0, b)$ be the y-intercept of the graph of L. Therefore,

$$
\begin{aligned}
\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix} \\
&= \begin{bmatrix} \frac{u_1}{\|\mathbf{u}\|} & \frac{-u_2}{\|\mathbf{u}\|} \\ \frac{u_2}{\|\mathbf{u}\|} & \frac{u_1}{\|\mathbf{u}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix} \\
&= \begin{bmatrix} \frac{u_1}{\|\mathbf{u}\|} & \frac{u_2}{\|\mathbf{u}\|} \\ \frac{-u_2}{\|\mathbf{u}\|} & \frac{u_1}{\|\mathbf{u}\|} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix},
\end{aligned}
\tag{17}
$$

which coincides with (4).

In summary, the above proof leads to:

Theorem 12 *Let S be the smooth parametric surface, $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \le t \le t_2, s_1 \le s \le s_2$. Let R be the region bounded by S, the plane $P : ax + by + cz = d$, and the perpendicular projection from the surface to the plane P. Then the volume of R is given by*

$$
\begin{aligned}
&\int\int r(s,t)\, dp\, dq \\
&= \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(s,t) \left| \begin{bmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{bmatrix} \right| ds\, dt,
\end{aligned}
\tag{18}
$$

where $\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} = [p_1, p_2, p_1 \times p_2]^{-1} \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) - \frac{d}{c} \end{bmatrix}$ *, and p_1 and p_2 are two orthonormal vectors on the plane P.*

For the remainder of the paper, the word "volume" represents a signed volume, which depends on the orientation of a surface. The Divergence Theorem in our discussion can be summarized as follows:

Theorem 13 *Let*

$$\mathbf{F}\left(x,y,z\right) = \left[\begin{array}{ccc} P\left(x,y,z\right), & Q\left(x,y,z\right), & R\left(x,y,z\right) \end{array}\right]$$

be a vector field defined on a solid region R (with $\mathrm{div}\,\mathbf{F} = \mathbf{1}$) bounded by a smooth surface $\mathbf{w}(s,t) = \{[x(s,t), y(s,t), z(s,t))] : t_1 \leq t \leq t_2, s_1 \leq s \leq s_2\}$, a plane $P : ax + by + cz = d$, and the perpendicular projection from the surface to the plane P. Let S be the boundary surface of R and $\mathbf{n}$ be the outward-pointing unit normal vector at each point of S. If the components of $\mathbf{F}$ have continuous first order partial derivatives on an open region that contains R, then

$$\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint\limits_{R} \mathrm{div}\,\mathbf{F}\,dV \tag{19}$$

$$= \;\; \text{the absolute value of } \int_{t_1}^{t_2}\int_{s_1}^{s_2} r(s,t) \left| \left[\begin{array}{cc} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{array} \right] \right| ds\,dt,$$

where $d\mathbf{S} = \mathbf{n}\,dS$, $\left[\begin{array}{c} p(s,t) \\ q(s,t) \\ r(s,t) \end{array}\right] = [p_1, p_2, p_1 \times p_2]^{-1} \left[\begin{array}{c} x(s,t) \\ y(s,t) \\ z(s,t) - \frac{d}{c} \end{array}\right]$, $\mathrm{div}\,\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$, and p_1 and p_2 are two orthonormal vectors on the plane P.

Remark 14 *Choose the plane to be the xy-plane (i.e., $z = 0$) and use the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$. Then (19) reduces to*

$$\int\int z(s,t)dx\,dy \tag{20}$$

$$= \int_{t_1}^{t_2}\int_{s_1}^{s_2} z(s,t) \left| \left[\begin{array}{cc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right] \right| ds\,dt,$$

as expected.

Corollary 15 *If the smooth surface $\mathbf{w}(s,t)$ is an orientable closed surface, (19) produces the volume of the region bounded by the closed surface, and this volume is independent of the location of the plane P.*

Remark 16 *This is a simple application of the Divergence Theorem.*

The next two examples demonstrate the use of Theorem 13 and Corollary 15.

Example 17 *Let S be the ellipsoid $\left[\begin{array}{c} x(s,t) \\ y(s,t) \\ z(s,t) \end{array}\right] = \left[\begin{array}{c} 3 + \sin t \cdot \cos s \\ 2 + 2\sin t \cdot \sin s \\ 3 + 3\cos t \end{array}\right]$, where $s \in [0, 2\pi]$, and $t \in$*

$[0, \pi]$, and P be the plane $2x + y + z = 0$ (passing through the origin). See Figure 6 below.
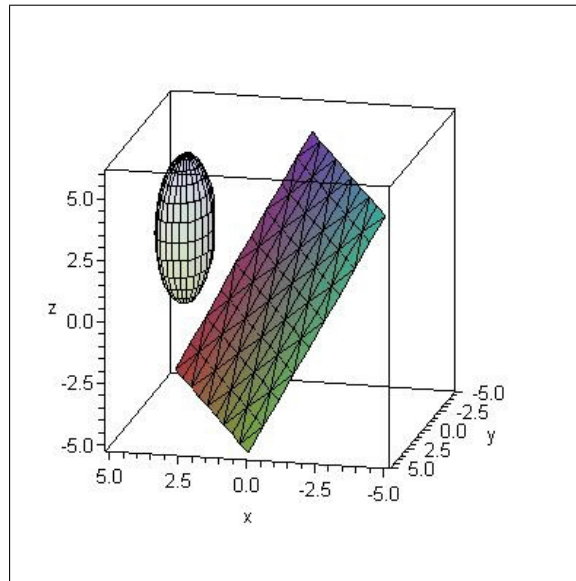

Figure 7. Volume of a surface

Then (18) produces the volume of the ellipsoid regardless of the location of the plane P. This is a direct observation from the Divergence Theorem. (See Maple worksheet in [10]).

Example 18 *Let S be the ellipsoid represented by the following parametric equation*

$$\begin{bmatrix} \cos(4) \cdot \sin(t) \cdot \cos(s) - 3\sin(4) \cdot \cos(t) \\ \sin(2) \cdot \sin(4) \cdot \sin(t) \cdot \cos(s) + 2\cos(2) \cdot \sin(t) \cdot \sin(s) + 3\sin(2) \cdot \cos(4) \cdot \cos(t) \\ \cos(2) \cdot \sin(4) \cdot \sin(t) \cdot \cos(s) - 2\sin(2) \cdot \sin(t) \cdot \sin(s) + 3\cos(2) \cdot \cos(4) \cdot \cos(t) \end{bmatrix},$$

where $s \in [\pi, 2\pi]$, and $t \in [0, \dfrac{\pi}{2}]$, and P be the plane $x + y + z = -3$. See the graph below.
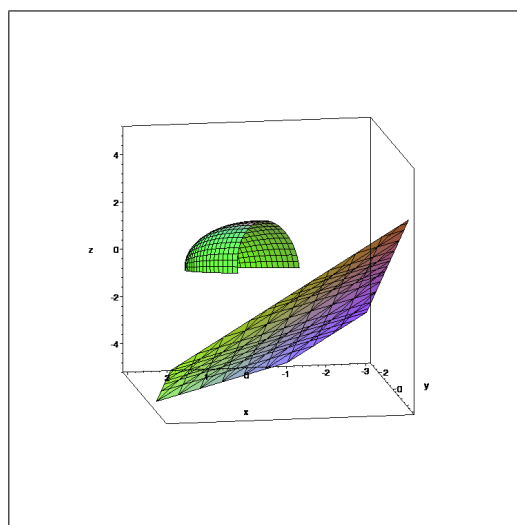

Figure 8. Volume of a surface with respect to a slanted plane

Let R be the region bounded by S, the plane P, and the perpendicular projection from the surface to the plane P. Then the signed volume of R is $-2\pi + \frac{26}{9}$ when Theorem 12 or 13 is applied. (See Maple worksheet 11.)

3   Another Rotation and Shifting Method

It may not be trivial to find the intersection between a smooth surface and a plane. Consequently, it is difficult to find the volume of the solid bounded by the surface and the plane. In this section, we describe how we use the idea of rotation to aid us in finding the volume of the solid bounded by a closed orientable smooth surface S of the form $F(x, y, z) = 0$ and a plane $P_1 : ax + by + cz = d$. Throughout this section, we assume the plane P_1 cuts the solid bounded by the surface S into only two pieces. Our goal is to find the volumes of these two respective pieces bounded by P_1 and S. The procedure is as follows:

Step 1. We first consider the plane $P_2 : ax + by + cz = 0$, which is parallel to P_1 and passes through the origin.

Step 2. We pick three points X_0, X_1, and X_2 on P_2, and form the orthonormal basis $\{p_1, p_2, n\}$, where n is the unit normal vector of P_2 (or P_1) that points toward the the region whose volume we seek.

Step 3. We now consider the plane $P_1 : ax + by + cz = d$. By Theorem 12, we have $X - \begin{bmatrix} 0 \\ 0 \\ \frac{d}{c} \end{bmatrix} = AX^*$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = [p_1, p_2, n]$, and $X^* = \begin{bmatrix} p \\ q \\ w \end{bmatrix}$. In other words, we express X in terms of X^*.

Step 4. We substitute X with $A \begin{bmatrix} p \\ q \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d/c \end{bmatrix}$ into the formula $F(x, y, z) = 0$, which results in an equation $G(p, q, w) = 0$.

Step 5. Employing spherical coordinates we solve the equation $G(p, q, w) = G(\rho \sin t \cos s, \rho \sin t \sin s, \rho \cos t) = 0$ for ρ.

Step 6. We use the following formula to find the volume of the region bounded by P_1 and $F(x, y, z) = 0$:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\int_0^{\rho(s,t)} \left(\rho^2 \sin t \right) d\rho \right) dt ds. \tag{21}$$

Note: Maple allows us to solve for ρ above. If the exact integration value from (21) is not possible when using a CAS, we apply higher dimension Simpson's rule to obtain an approximation. We summarize the discussion above into the following:

Theorem 19 *Let S be the smooth parametric surface of the form $F(x, y, z) = 0$ (or we write $F(X) = 0$ for simplicity). Let R be the region bounded by S and the plane $P : ax + by + cz = d$. Furthermore, we assume the following conditions are met:*

(i) If $G(p, q, w) = F\left(A\begin{bmatrix} p \\ q \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d/c \end{bmatrix}\right) = 0$, where $A = [p_1, p_2, n]$ and p_1, p_2 are two perpendicular unit vectors on the plane P, and n is the unit normal vector of P that points toward the the region whose volume we seek.

.

(ii) If $G(\rho \sin t \cos s, \rho \sin t \sin s, \rho \cos t) = 0$ is solvable for ρ,
then the volume of R is given by

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\int_0^{\rho(s,t)} (\rho^2 \sin t) \, d\rho \right) dt ds. \tag{22}$$

3.1 An Example

Example 20 *Given the surface $S : 4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz - 5 = 0$ and the plane $P :$ $x + y + z - 1 = 0$, find the respective volumes bounded by the ellipsoid and the plane. The graph is shown below.*
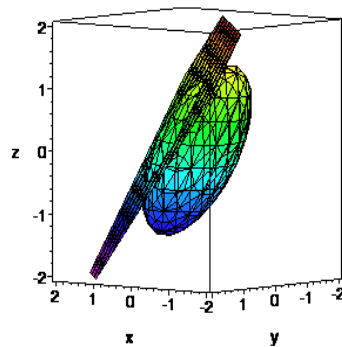

Figure 9. Volume bounded by a surface and a slanted plane

Step 1. We pick three points $X_0 = (0, 0, 0)$, $X_1 = (1, -1, 0)$, and $X_2 = (1, 0, -1)$ on the plane $P_2 : x + y + z = 0$ to form the vectors $u = X_1 - X_0$ and $v = X_2 - X_0$. We form orthonormal vectors

$$p_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{-\sqrt{2}}{2} \end{bmatrix}, p_2 = \begin{bmatrix} \frac{-\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{6}}{6} \end{bmatrix}, and \ n = p_1 \times p_2 = \begin{bmatrix} \frac{\sqrt{12}}{6} \\ \frac{\sqrt{12}}{6} \\ \frac{\sqrt{12}}{6} \end{bmatrix}. \tag{23}$$

Step 2. We compute $X = AX^*$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = [p_1, p_2, n]$, and $X^* = \begin{bmatrix} p \\ q \\ w \end{bmatrix}$.

Step 3. We substitute X with $A\begin{bmatrix} p \\ q \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into the formula $F(x, y, z) = 0$, or $4x^2 + 4y^2 + 4z^2 +$

$4xy + 4xz + 4yz - 5 = 0$, *to get an equation* $G(p, q, w) = 0$.

Step 4. Solve $G(\rho \sin t \cos s, \rho \sin t \sin s, \rho \cos t) = 0$ *for* ρ*, for which we obtained two values from Maple:*

Case 1. If we use the following (see Maple worksheet, Section3-Example20.mws)

$$
\begin{aligned}
\rho = \ & 1/6 * (-8 * 3\,\hat{}\,(1/2) * cos(t) + 3 * 2\,\hat{}\,(1/2) * sin(t) * cos(s) + 3\,\hat{}\,(1/2) * 2\,\hat{}\,(1/2) * sin(t) * sin(s) + \\
& 2 * (60 * cos(t)\,\hat{}\,2 - 12 * sin(t) * cos(s) * 3\,\hat{}\,(1/2) * 2\,\hat{}\,(1/2) * cos(t) \\
& -12 * sin(t) * sin(s) * 2\,\hat{}\,(1/2) * cos(t) \\
& +3 * cos(s)\,\hat{}\,2 - 3 * cos(s)\,\hat{}\,2 * cos(t)\,\hat{}\,2 + 3 * cos(s) * 3\,\hat{}\,(1/2) * sin(s) - \\
& 3 * cos(s) * 3\,\hat{}\,(1/2) * sin(s) * cos(t)\,\hat{}\,2 + 6)\,\hat{}\,(1/2))/(1 + 3 * cos(t)\,\hat{}\,2),
\end{aligned}
\tag{24}
$$

the volume of one portion of the region bounded by P_1 *and* $F(x, y, z) = 0$ *is*

$$
\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\int_0^{\rho(s,t)} \left(\rho^2 \sin t \right) d\rho \right) dt\,ds = .4110463566,
$$

when we use 200 points Simpson's Rule for approximation.

Case 2. If we use the following

$$
\begin{aligned}
\rho = \ & 1/6 * (-8 * 3\,\hat{}\,(1/2) * cos(t) + 3 * 2\,\hat{}\,(1/2) * sin(t) * cos(s) + 3\,\hat{}\,(1/2) * 2\,\hat{}\,(1/2) * sin(t) * sin(s) \\
& -(246 * cos(t)\,\hat{}\,2 - 48 * sin(t) * cos(s) * 3\,\hat{}\,(1/2) * 2\,\hat{}\,(1/2) * cos(t) \\
& -48 * sin(t) * sin(s) * 2\,\hat{}\,(1/2) * cos(t) \\
& +18 * sin(t)\,\hat{}\,2 * cos(s)\,\hat{}\,2 + 12 * sin(t)\,\hat{}\,2 * cos(s) * 3\,\hat{}\,(1/2) * sin(s) + 6 * sin(t)\,\hat{}\,2 * sin(s)\,\hat{}\,2 \\
& +18)\,\hat{}\,(1/2))/(1 + 3 * cos(t)\,\hat{}\,2),
\end{aligned}
\tag{25}
$$

the volume of the other portion of the region bounded by P_1 *and* $F(x, y, z) = 0$ *is*

$$
\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\int_0^{\rho(s,t)} \left(\rho^2 \sin t \right) d\rho \right) dt\,ds = -7.867777159
\tag{26}
$$

when we use 200 *points Simpson's Rule for approximation. Note that the total volume, by taking the absolute value, is about* 8.278823557*, which is accurate up to* 8 *decimal places. (See Maple worksheet [12].) It can be shown that the ellipsoid can be rotated to the form of*

$$
\frac{x^2}{5/2} + \frac{y^2}{5/2} + \frac{z^2}{5/8} = 1,
$$

which has a volume of 8.278823556*.*

An Exercise. We are given the surface $S : x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and the plane $P : x + y + z = \sqrt{3}$. Find the respective volumes bounded by the ellipsoid and the plane.

Answer: we get two values, 4.463982383 and -20.66875886, when we use $n = 200$ in Simpson's rule. (See Maple worksheet [13].) Readers can verify that the sum of the absolute values adds up to the volume of an ellipsoid by using the formula $\frac{4}{3}\pi abc$, when the ellipsoid is written as $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Remark 21 *We note that the rotation method described here applies to the smooth surface satisfying* $F(x, y, z) = 0$ *and the method is valid only when* ρ *is solvable by a CAS.*

4   Conclusion

In an ordinary calculus textbook, the Divergence Theorem and Green's Theorem are usually taught in the last semester of a calculus sequence, to a group of students who completed some basic training in calculus. These theorems are important and yet difficult for many students. In this paper, we describe special ways of finding net *signed* areas and volumes through smooth parametric curves and surfaces.

The methods discussed in Sections 2 and 3 provide us with a way to find the net signed volume. The methods described here are accessible to those students who have knowledge in 2-D and 3-D Riemann integration. In particular, formulae (7) and (19) are natural extensions when we replace the graphs represented by $y = f(x)$ and $z = f(x, y)$ by a parametric curve or surface with respect to a line $y = mx + b$ or a plane $ax + by + cz = d$, respectively. Consequently, the contents are accessible to a wider audience and can be used to prepare students for the study of the Divergence Theorem, Green's Theorem, and even Gauss-Bonnet Theorem in Differential Geometry.

Note that the results obtained in this paper are not possible without the help of technology. Dynamic Geometry collects the data, estimates the area, and allows us to construct the animation. The CAS aids us in the computation, graphing, and verification of conjectures. Authors anticipate that mathematical content will evolve as technological tools advance.

References

[1] Stewart, J. (2001), Calculus-Concepts and Contexts, second edition, Brooks/Cole Publishing, ISBN 0-534-37718-1.

[2] Larson, R.,Hostetler, R., Edwards, B., Essential Calculus-Early Transcendental Functions, Houghton Miffin Company, ISBN 13: 978-0-618-87918-2.

Software Packages

[3] [ClassPad] A product of CASIO Computer Ltd., http://classpad.net or http://classpad.org/.

[4] [Maple] A product of Maplesoft, http://www.maplesoft.com/.

Supplemental Electronic Materials

[5] Yang, W.-C., *a video clip* which summarizes how ClassPad Manager is used in Section 1.1.

[6] Yang, W.-C., *ClassPad eActivity* used in Section 1.1.

[7] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 7.

[8] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 10.

[9] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 11.

[10] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 17.

[11] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 18.

[12] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 20.

[13] Yang, W.-C., Lo, M.-L., *Maple worksheet* for An Exercise in Section 3.